1 Antiderivatives and differential equations

Given $f(x)$ find $F(x)$ such that $F'(x) = f(x)$. $F(x) + C$ is the *general antiderivative* of $f(x)$.

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$cosx$	$sinx$
$sinx$	$-cosx$
$sec^2 x$	$tanx$
$secxtanx$	$secx$
$\frac{1}{x}$	$ln\|x\|$

A *differential equation* is an equation involving one or more derivatives of an unknown function that we want to find.

We find a *particular solution* by imposing initial conditions of the form $F(a) = b_0$, $F'(a) = b_1$, $F''(a) = b_2$,...

2 Example

Find the height of a bull above the surface of the earth at time t if it was projected upwards at time $t = 0$ from a height of $2m$ with velocity $19.6m/s$

The differential equation that governs height h is $\frac{d^2h}{dt^2} = -9.8m/s^2$.

$\frac{dh}{dt} = -9.8t + C$.

$\frac{dh}{dt}$ (velocity) $= 19.6$ at $t = 0$).

so $\frac{dh}{dt} = -9.8t + 19.6$.

$h = -4.9t^2 + 19.6t + C_2$ ($h = 2$ at $t = 0$).

so $h = -4.9t^2 + 19.6t + 2$.

General formula: $h = -4.9t^2 + V_0 h + h_0$.

$V_0 =$ initial velocity

$h_0 =$ initial height.

3 Estimating Area - the Definite Integral

The area A of the region S that lies under the graph of the continuous function f is the limit of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

4 Definite integral

If i is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a)$, x_1, x_2, ...,

$x_n(= b)$ be the endpoints of these subintervals and we choose sample points x_1^*, x_2^*, ..., x_n^* in these subintervals, so $x_i^*)$ lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1} nf(x_i^*)\Delta x$$

$\int$ is the integral sign; $f(x)$ is the integrand. a and b are the lower and upper limits of integration. This procedure is known as integration.

5 Riemann's Sum

$$\sum_{i=1} nf(x_i^*)\Delta x$$

(General)

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

replace $\lim \sum$ with $\int$
x_i^* with x
Δx with dx.

x_i^* may be the:
right endpoint $\to R_{\text{max}}(x_i)$
left endpoint $\to R_{\text{min}}(x_{i-1})$
midpoint $[x_{i-1}, x_i] \to R_{\text{mid}}(\overline{x_i})$
$R_{\text{min}} \leq R_{\text{general}} \leq R_{\text{max}}$
$\lim_{n\to\infty} R_{\text{min}} = \lim_{n\to\infty} R_{\text{general}} = \lim_{n\to\infty} R_{\text{max}}$

6 Evaluating Integrals

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
$\sum_{i=1}^{n} c = nc$
$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_1 + \sum_{i=1}^{n} b_i$
$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_1 - \sum_{i=1}^{n} b_i$

Riemann's Sum contains terms $\frac{b-a}{n} f(x_i^*)$ for $f(x_i^*) < 0$
It represents the area of the rectangles above the x-axis minus the area of the rectangles below the x-axis.

Examples:
$\int_0^1 \sqrt{1 - x^2}dx$

$y = \sqrt{1-x^2}$; $y^2 = 1 - x^2$; $x^2 + y^2 = 1$ (ie - the function of a quarter of a circle. Thus it's integral is the area of a quarter of a circle)
$\int_1^0 \sqrt{1-x^2} = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$

7 Properties of the Definite Integral

$\int_b^a f(x)dx = -\int_a^b f(x)dx$
$\int_a^a f(x)dx = 0$
$\int_a^b cdx = c(b-a)$ (c is a constant)
$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
$\int_a^b cf(x)dx = c\int_a^b f(x)dx$
$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

8 Fundamental theorem of calculus

8.1 Indefinite Integration

Suppose f is continuous on $[a,b]$.
If $g(x) = \int_a^b f(t)dt$, then $g'(x) = f(x)$.
$\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We now denote the *general antiderivative* of $f(x)$ by $\int f(x)dx$ (no limits); we call this the *indefinite integral* of $f(x)$.

$\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

The process of finding it is *indefinite integration.*

A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function.

Table of indefinite integrals

$\int cf(x)dx$	$= c\int f(x)dx$
$\int [f(x)+g(x)dx$	$= \int f(x)dx + \int g(x)dx$
$\int kdx$	$= kx + C$
$\int x^n dx$	$= \frac{x^{n+1}}{n+1} + C(n \neq -1)$
$\int sin(x)dx$	$= -cosx + C$
$\int cos(x)dx$	$= sinx + C$
$\int sec^2(x)dx$	$= tanx + C$
$\int csc^2(x)dx$	$= -cotx + C$
$\int sec(x)tan(x)dx$	$= secx + C$
$\int csc(x)cot(x)dx$	$= -cscx + C$
$\int \frac{1}{u}du$	$=$
$\int e^u du$	$=$
$\int a^u du$	$=$

$$\int F'(x)dx = F(b) - F(a)$$

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

The substitution rule for definite integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = f(x)$ then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Integrals on symmetric functions:

Suppose f is continuous on $[-a, a]$.

(a) If f is even $(f(-x) = f(x))$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(b) If f is odd $(f(-x) = -f(x))$, then $\int_{-a}^{a} f(x)dx = 0$.

9 Numerical Integration

9.1 Rectangular Midpoint Rule

$\int_a^b f(x)dx \approx M_n = \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + \ldots + f(\overline{x_n})]$ where $\Delta x = \frac{b-a}{n}$ and $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$

9.2 Trapezoidal Rule

$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(n-1) + f(n)]$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Error Bounds

Suppose $|f''(x)| \leq k$ for $a \leq x \leq b$

$|E_T| \leq \frac{K(b-a)^3}{12n^2}$

$|E_M| \leq \frac{K(b-a)^3}{24n^2}$

9.3 Simpson's Rule

$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$

where n is even and $\Delta x = \frac{b-a}{n}$

Error Bound

Suppose that $|f^{(n)}(x) \leq K$ for $a \leq x \leq b$

$|E_S| \leq \frac{K(b-a)^5}{180n^4}$

10 Areas of General Regions

The area A of a region bounded by two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ where f and g are continuous and $f(x) \geq g(x) \forall x$ in $[a.b]$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]dx$$

If we are to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x and $g(x) \geq f(x)$ for other values of x then we split the given region into several regions and sum the absolute values of the areas for these regions.
Some regions are best treated as retarding x as a function of y.

$$A = \int_c^d [f(y) - g(y)]dy$$

11 Exponential and Logarithmic Functions

An exponential function is a function of the form $f(x) = a^x$ where a is a positive constant. If a is positive then $a^x = \lim_{r\to x} a^r$ (where r is rational)

Laws of Exponents
$a^{x+y} = a^x a^y$
$a^{x-y} = \frac{a^x}{a^y}$
$(a^x)^y = a^{xy}$
$(ab)^x = a^x b^x$

If $a > 1$ then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$
If $0 < a < 1$ then $\lim_{x\to\infty} a^x = 0$ abd $lim_{x\to-\infty} = \infty$

11.1 Derivatives of Exponential Functions

$f(x) = a^x$
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0} \frac{a^h-1}{h}$

$$\lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)$$

$f'(x) = a^x f'(0)$
for $a = 2$, $f'(0) = \lim_{h\to 0} \frac{2^h-1}{h} \approx 0.69$

11.2 Definition of the Number e

e is the number s.t. $lim_{h\to 0}\frac{a^h-1}{h} = 1$
$\frac{d}{dx}(e^x)$
$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

11.3 Properties of the exponential function

The exponential function $f(x) = e^x$ is an increasing continuous function with domain R and range $0, \infty)$. Thus $e^x > 0, \forall x$.

$$\lim_{x\to -\infty} e^x = 0$$

$$\lim_{x\to \infty} e^x = \infty$$

So the x axis is a horizontal asymptote of $f(x) = e^x$.

$$\int e^x dx = e^x + C$$

12 Logarithmic function with base a

$log_a x = y \Leftrightarrow a^y = x$
$log_a a^x = x \forall x \in R$
$a^{log_a x} = x \forall x > 0$
$log_a(xy) = log_a x + log_a y$
$log_a(\frac{x}{y}) = log_a x - log_a y$
$log_a(x^n) = nlog_a x$
if $a > 1$, then

$$\lim_{x\to \infty} log_a x = \infty$$

and

$$\lim_{x\to 0^+} log_a x = -\infty$$

13 Natural logarithms: $log_e x = lnx$

$lnx = y \Leftrightarrow e^y = x$
$ln(e^x) = x, x \in R$
$e^{lnx} = x, x > 0$
$lne = 1$
For any positive number a, $(a \neq 1)$, $log_a x = \frac{lnx}{lna}$.

$$\lim_{x\to \infty} lnx = \infty$$

and

$$\lim_{x\to 0^+} lnx = -\infty$$

14 Derivatives of logarithmic functions

$\frac{d}{dx}(ln|x|) = \frac{1}{x}$
$\frac{d}{dx}(lnx) = \frac{1}{u} \cdot \frac{du}{dx}$
$\int \frac{1}{x} dx = ln|x| + C$
$\frac{d}{dx}(log_a x) = \frac{1}{xlna}$