MA1010 - Continuous Mathematics

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Chapter 1

Introduction to Calculus

1.1 Constant velocity

We begin with the fundamental problems that calculus was designed to solve. Suppose we make a journey in a car and observe the speedometer reading v for velocity and the odometer reading f for distance from the start. The question is: if we only have one of these readings can we recover the other? Finding the velocity from a record of the distance (from the start) at each time instant is the process of *differentiation* - the central idea of *differential calculus*. Finding the distance (from the start) from a record of the velocity is the process of *integration* - the central idea of *integral calculus*.

Suppose the velocity is fixed at $v = 60$ (km per hour). Then f increases at this constant rate. After two hours, $f = 120$ (km); after four hours, $f = 240$ (km); after t hours, $f = 60t$ (km). We say that f increases linearly with time since the graph of f against t is a straight line. Notice that since both the velocity and the distance depend on the time we can say that they are "functions of time" and write $v(t)$ and $f(t)$ for their values at a time t.

In this example the distance started at 0. If the distance had started at 20, say, then we would obtain the same graph raised by 20 units (km) throughout. Another alternative would be to drive the car backwards at say 30 (km per hour), that is, take $v = -30$. Then the distance would be negative. Note that in this case we have to assume the odometer runs backwards, because f represents distance from the starting point or origin, or more precisely, position.

It is clear that geometrically the velocity is the slope of the distance graph

$$slope = \frac{\text{change in distance}}{\text{change in time}} = \frac{vt}{t} = v$$

Apparently, the opposite of slope is area so that the distance is the area under the velocity graph. This is not so obvious, but another example helps. If we go forward at constant velocity V and then backwards for the same length of time at velocity $-V$, then our final destination is at distance 0 from the start.

The total area between the graph of V and the t-axis is $2V + (-2V) = 0$ (the idea of negative area will become more clear later).

1.2 Instantaneous change of velocity

In this example the function $V(t)$ is defined as

$$V(t) = \begin{cases} V & \text{if } 0 \leq t < 2 \\ ? & \text{if } t = 2 \\ -V & \text{if } 2 < t \leq 4 \end{cases}$$

Of course this is not very realistic, because a vehicle cannot suddenly start up at V miles per hour, nor suddenly change from going forwards at V to backwards at V. The first problem is easily resolved - we can imagine the car travelling at V past a fixed reference point at which we start the clock at $t = 0$. However, the second is not as easy - here the value of V at $t = 2$ is undefined and its graph has a *discontinuity.* The graph of f is *piecewise linear* with the formula

$$f(t) = \begin{cases} Vt & 0 \leq t \leq 2 \\ V(4-t) & 2 \leq t \leq 4 \end{cases}$$

Notice that these two formulae coincide for $t = 2: f(2) = V(2) = V(4-2)$, so that f is "continuous". However, there is no slope at the sharp point of the graph, and this corresponds to the discontinuity in the velocity function.

1.3 Continuous change of velocity

Now suppose the velocity changes over time - the car accelerates or decelerates. For example, suppose that $V(t) = 2t$. If we measure t in seconds and V in metres per second, the distance is then in metres. After $10s$ the velocity is $20m/s$ and after $15s$ the velocity is $30m/s$. Clearly the car is covering more ground in each successive second (in fact the acceleration is $2m/s$ per second, or $2m/s^2$); but the question is how far it has gone.

We answer this in a roundabout way by considering the distance function $f(t) = t$. The graph of this function is a *parabola.* At $t = 5$ the distance is 25 and at $t = 10$ the distance is 100. The average velocity over the first $10s$ is $\frac{100}{10} = 10m/s$. The average velocity over the first $11s$ is $\frac{121}{11} = 11m/s$.

But what is the velocity at $t = 10$? The average velocity in the second between $t = 10$ and $t = 11$ is $\frac{f(11) - f(1)}{11-10} = \frac{121-100}{1} = 21$. Clearly, the average velocity over an interval between two time instants $t = t_1$ and $t = t_2$ is $\frac{f(t_2)-f(t_1)}{t_2-t_1}$. In different notation, the average velocity between time t and $t + h$ is $\frac{f(t+h)-f(t)}{h}$. We can then compute the average velocity over the half second from $t = 10$ to $t = 10.5$ as $\frac{f(10.5)-f(10)}{0.5} = \frac{110.25-100}{0.5} = 20.5$. The way to find the instantaneous velocity at $t = 10$ is to keep reducing the time interval. In general the average velocity between $t = 10$ and $t = 10 + h$ is $\frac{f(10+h)-f(10)}{h} = \frac{(10+h)^2-10^2}{h} = \frac{100+20h+h^2-100}{h} = \frac{(20+h)h}{h} = 20 + h$. This fits with the earlier calculations. If we reduce h to $\frac{1}{1000000}s$ then the average is $20 + \frac{1}{1000000}m/s$. The conclusion is that the velocity at $t = 10$ is 20. Notice that this is the same value as that on the V curve $V(10) = 20$.

Now for any t we can compute the velocity at t from $V_{average} = \frac{f(t+h)-f(t)}{h} = \frac{(t+h)^2-t^2}{h} = \frac{t^2+2th+h^2-t^2}{h} = 2t + h$.

If we take smaller and smaller intervals h, this approaches the value $2t$, and thus we see that the function $V = 2t$ represents the velocity when $f = t^2$ represents the distance. This is the essential idea of *differentiation*.

The other direction is given by the Fundamental Theorem of Calculus which says that if $V = 2t$ represents the velocity then the area under the curve gives the distance. It is obvious that this area is the area of a triangle with base t and height $2t$ so its area is $\frac{1}{2}(t)(2t) = t^2$. Thus $f(t) = t^2$ when $V(t) = 2t$. This is the essential idea of *integration*.

From what we have seen the process of differentiation determines the *slope of the tangent to the curve* $f(t)$. Thus the *slope of f(t)* is given by $V(t)$.

1.4 Examples

Example: if $f(t) = D$ then

$$f'(t) = \lim_{\Delta t \to 0} \frac{D - D}{\Delta t} = \lim_{\Delta t \to 0} \frac{0}{\Delta t} = \lim_{\Delta t \to 0} 0 = 0$$

.

Example: Constant velocity V: the distance if $f(t) = Vt$. The derivative is

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{V(t + \Delta t) - Vt}{\Delta t} = V$$

Example: Constant velocity V until $t = 2$ and $-V$ thereafter. Up to time $t = 2$ we still have $f(t) = V(t)$, but then, if we find the velocity at $t = 2$ by taking intervals before $t = 2$ we get slope V while after $t = 2$ we get slope $-V$. These are incompatible so we say the derivative at $t = 2$ does not exist or is undefined.

Example: The demand function is $r(p) = \frac{1}{p}$. High price means low demand. We want to know: if we increase the price, how quickly does the demand fall? $r(p) = \frac{1}{p}$.

$$\frac{dr}{dp} = \lim_{\Delta p \to 0} \frac{r(p + \Delta p) - r(p)}{\Delta p} = \lim_{\Delta p \to 0} \frac{\frac{1}{p+\Delta p} - \frac{1}{p}}{\Delta p}$$

$$= \lim_{\Delta p \to 0} \frac{\frac{-\Delta p}{p(p+\Delta p)}}{\Delta p} = \lim_{\Delta p \to 0} \frac{-1}{p(p + \Delta p)} = \frac{-1}{p^2}$$

Thus demand falls as the inverse square of the price.

Chapter 2

Derivatives

For an arbitrary function $f(t)$ we define the derivative $f'(t)$ by

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Here we have replaced V by $f'(t)$ because our function f might not be distance and therefore its derivative will not be velocity. Also, we have replaced h with Δt which can be regarded as the *change in* t. To make things symmetrical we also represent the change in f by $\Delta f = f(t + \Delta t) - f(t)$. The *limit of* $\frac{\Delta f}{\Delta t}$ *as* Δt *approaches* 0 *is the derivative, if this limit exists*:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}$$

.

It is important to realise that the limiting process is not just setting $\Delta t = 0$. This would be meaningless. We must have Δt and Δf approaching zero together. Moreover, we can compute the slope of f at time t by using a sequence of intervals *before* t. It should turn out that we get the same result, but sometimes this does ot happen and in those cases the derivative does not exist, i.e. the limit

$$\lim_{\Delta t \to 0}$$

is not defined.

Generally speaking when we want to consider functions in the abstract i.e. divorced from any particular application we use x, y for the independent and dependent variables respectively. Often t is used as an independent variable and letters f, g, h, u, v, y are used for functions. The derivative of a function is its *rate of change with respect to the corresponding independent variable.*

Here is an early example but a nice one. Suppose we have a function $u(x)$ so its derivative is $\frac{du}{dx}$. Now consider the function $f(x) = (u(x))^2$. What is the derivative of $f(x)$? Well, proceed routinely to compute

$$\Delta f = (u(x + \Delta x))^2 - (u(x))^2 = [u(x + \Delta x) + u(x)][u(x + \Delta x) - u(x)]$$

using the difference of two squares. Now

$$\frac{\Delta f}{\Delta x} = [u(x+\Delta x) + u(x)]\left[\frac{u(x+\Delta x) - u(x)}{\Delta x}\right]$$

and this approaches $2u(x)\frac{du}{dx}$ as Δx approaches 0.

Thus, if $u = x^2$, we see that the derivative of $f(x) = x^4$ is $2(x^2)(2x) = 4x^3$; if $u = \frac{1}{x}$ then the derivative of $f(x) = \frac{1}{x^2}$ is $2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{-2}{x^3}$.

What we need now is a longer list of functions and derivatives.

2.1 Examples

Example: Let $f(x) = 5x^3$. Then

$$\frac{\Delta f}{\Delta x} = \frac{5(x+h)^3 - 5x^3}{h} = 5\left[\frac{(x+h)^3 - x^3}{h}\right] = 5(3x^2) = 15x^2$$

so $f'(x) = 15x^2$.

Example: Let $u(x) = x^2 + x^3$. Then

$$\frac{\Delta u}{\Delta x} = \frac{[(x+\Delta x)^2 + (x+\Delta x)^3] - [x^2 + x^3]}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 - x^2}{\Delta x} + \frac{(x+\Delta x)^3 - x^3}{\Delta x} = 2x + 3x^2$$

Example: Let $f(x) = \sqrt{x}$. Then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

Recall that the binomial theorem gives the expansion $(x+h)^n = x^n + nx^{n-1}h + \ldots + h^n$ and every term after $nx^{n-1}h$ is divisible by h^2 provided $n \geq 3$.

For example, with $n = 3$ we have $x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

Thus, if $f(x) = x^n$ then

$$\frac{\Delta f}{\Delta x} = \frac{(f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$= \frac{nx^{n-1}h + \ldots + h^n}{h} = nx^{n-1} + h(\ldots)$$

so

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = nx^{n-1}$$

. Here we have used the letter x as an alternative to Δx because it makes the computation easier to follow.

Thus $\frac{df}{dx} = nx^{n-1}$ when $f(x) = x^n$ for $n \geq 3$. But this also fits the cases $n = 1, 2$ since if $f(x) = x$ then $\frac{df}{dx} = 1$ and if $f = x^2$, $\frac{df}{dx} = 2x$.

It is actually true that this rule extends to all powers of x so that $f(x) = x^r$ has derivative $f'(x) = rx^{r-1}$. Thus, as we saw earlier, the derivative of x^{-1} is $-x^{-2}$ and of x^{-2} is $-2x^{-3}$. Similarly, the derivative of $\sqrt(x) = x^{\frac{1}{2}}$ is $\frac{1}{2}x^{-\frac{1}{2}}$ as we shall see later.

Now suppose we consider multiplying the function $f(x)$ by a constant c. Then writing $g(x) = cf(x)$ we get:

$$\frac{\Delta g}{\Delta x} = \frac{cf(x + \Delta x) - cf(x)}{\Delta x} = c\left[\frac{f((x + \Delta x) - f(x)}{\Delta x}\right]$$

and this approaches $c\frac{df}{dx}$. Thus $g'(x) = cf'(x)$.

Also, consider adding two functions f, g to get $u(x) = f(x) + g(x)$. Then

$$\frac{\Delta u}{\Delta x} = \frac{(f(x + \Delta x) + g(x + \Delta x) - (f(x) + g(x))}{\Delta x}$$

$$= \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

which approaches $\frac{df}{dx} + \frac{dg}{dx}$. Thus $u'(x) = f'(x) + g'(x)$.

In particular, if $u(x)$ is a polynomial function $u(x) = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n$ then $u'(x) = u_1 + 2u_2 x + \ldots + nu_n x^{n-1}$.

The process opposite to differentation involves solving a *differential equation* such as $\frac{dy}{dx} = 3x^2$. Here the derivative is known but the function y is not. We know that if $y = x^3$ then $\frac{dy}{dx} = 3x^2$, but we must be careful because if $y = x^3 = C$ for any constant C then also $\frac{dy}{dx} = 3x^2$. But notice that many differential equations have the same solution: e.g. $\frac{dy}{dx} = \frac{3y}{x}$, $\frac{dy}{dx} = 3y^{\frac{2}{3}}$.

Example: $y = x^4 - x^2 + 3$. At $x = 1$ we have $y = 3$. The slope in general is $\frac{dy}{dx} = 4x^3 - 2x$ so at $x = 1$ the slope is 2. Thus $f'(1) = 2$. Thus the tangent at $(1, 3)$ is $y - 3 = 2(x - 1)$. The normal at $(1, 3)$ is $y - 3 = -\frac{1}{2}(x - 1)$.

Example: You are on a roller-coaster whose track follows $y = x^2 + 4$. You see a friend at $(0, 0)$ and want to get there quickly. Where do you step off? Here $f(x) = x^2 + 4$, $f'(x) = 2x$.

The tangent at $x = a$ is $y - (a^2 + 4) = 2a(x - a)$ and if this passes through $(0, 0)$ then $-(a^2 + 4) = 2a(-a)$ which gives $a = \pm 2$. Step off at $(2, 8)$.

Example: Suppose you are driving up to a set of traffic lights and you know that it is less costly in fuel consumption to slow down and accelerate than to stop and start again. Your distance from the lights is $72m$ and a waiting car, which you cannot pass, will accelerate at $3m/s^2$ (i.e. with $V = 3t$. The lights will change in $4s$. Calculate the speed V at which you should travel so that you just catch the car ahead.

equal distances gives $T = \frac{1}{3}V + 4$.

equal distances gives $\frac{3}{2}(T - 4)^2 + 72 = VT$.

$\frac{3}{2}(\frac{1}{3}V+4-4)^2+72=V(\frac{1}{3}V+4)$
$V^2+24V-432=0$
$(V-12)(V+36)=0$ so $V=12m/s$ and $T=8s$.
Q: what speed should you adopt if there is no waiting car?

Example: in most epidemics the rate of growth of the number N of cases is proportional to the number of cases so $\frac{dN}{dt}=kN$ for some constant k. Later we will see that this gives $N=e^{kt}$ so the growth is *exponential*. However, there is always a limiting factor e.g. the epidemic may cause deaths and thus a limit to the population, or people who have recovered from the disease may no longer be susceptible, so the growth rate is not unbounded. However, for the AIDS epidemic in the US the number of cases to 1989 obeyed the following equation to within 2% : $N=174.6(t-1981.2)^3+340$ so we have approximately $\frac{dN}{dt}=\frac{3N}{t}$. In the years after 1989 the dependence of N on t dropped to quadratic from cubic.

2.2 Slope and the tangent line

In general for the function $y=f(x)$ the tangent at $x=a$ is given by $y-f(a)=f'(a)(x-a)$.

2.3 The normal line

In general for the function $y=f(x)$ the normal (perpendicular) at $x=a$ is given by $y-f(a)=-\frac{1}{f'(a)}(x-a)$.

2.4 The secant line

This is the line passing through the points $(t,f(t))$, $(t+\Delta t,f(t+\Delta t))$. Its slope is $\frac{f(t+\Delta t)-f(t)}{\Delta t}$ so its equation is $y-f(t)=\left(\frac{f(t+\Delta t)-f(t)}{\Delta t}\right)(x-a)$. Clearly, the secant line approaches the tangent line as Δt approaches zero.

Example: Let $y=\frac{3}{x}$ and find the secant at $(1,3)$. Slope is $\frac{\frac{3}{a}-3}{a-1}=\frac{3(1-a)}{a(a-1)}=-\frac{3}{a}$ (since $a\neq 1$). Thus secant is $y-3=-\frac{3}{a}(x-1)$.

2.5 Answers to Exercises 1

1. $-\frac{12}{x^2}$; $y-6=3(x-2)$; $y-6=\frac{1}{3}(x-2)$; $y-6=-\frac{3}{2}(x-2)$
2. $y-2=3(x-1)$; $y-2-\frac{1}{3}(x-1)$; $y-2=(h+3)(x-1)$
3. $y+1=3(x-1)$; $y=3x-4$
4. $y=6x$; no.
5. $y=x$; $(3,3)$

6. $x = y^2 \Rightarrow 1 = 2y\frac{dy}{dx} \ldots y - 2 = \frac{1}{4}(x-4)$

7. $y - a = (c+a)(x-a)$; $y - a = 2a(x-a)$

8. e.g. $y = 2x^3 - 9x^2 + 18x + 6$ from $\frac{dy}{dx} = (x-1)(x-2)+1$; $x = \pm 1$

9. $y = \frac{1}{5}x^2 + 2$; $y - 7 = -\frac{1}{2}(x-5)$

10. $\frac{a-c}{ac(c-a)} \to -\frac{1}{a^2}$; $y - \frac{1}{a} = -\frac{1}{a^2}(x-a)$

11. $y = 1$; $x = \frac{\pi}{2}$

12. $f'(b)$

13. $y - \frac{1}{a} = -\frac{1}{a^2}(x-a)$; $y = \frac{2}{a}$; $x = 2a$; 2

14. parallel; false unless tangents are horizontal.

15. $c = 4$; tangent at $x = 2$

16. $c = 5 \pm 2\sqrt{7}$

17. $(-3, 19)$; $(\frac{1}{3}, \frac{13}{27})$

18. e.g. slopes equal at $x_1 = \frac{2}{c}$, but then on line $y = \frac{8}{c}$ whereas on curve $y = \frac{4}{c}$

19. $c = 4$, $y = 3 - x$, tangent at $x = 1$

20. $b = -3$, $c = 2$, $d = 1$

21. $(1+h)^3$; $3h + 3h^2 + h^3$; $3 + 3h + h^2$; 3

22. can't succeed with a quadratic; $y = x^3 - 5x^2$ works.

23. parallel; the same.

24. (use 23) normal is both $y = -\frac{1}{2a}x + \frac{1}{2} + a^2 - 1$ and $y = -\frac{1}{1-2c}x + \frac{c}{1-2c} + c - c^2$ so compare coefficients.

25. $y = 2ax - a^2$; $Q = (0, -a^2)$; distance $a^2 + \frac{1}{4}$; angle of incidence equals angle of reflection.

26. $\sqrt{2}$ (the tangent has slope -1)

Chapter 3

Review of trigonometry

$cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$, $tan\theta = \frac{y}{x}$
$sec\theta = \frac{r}{x}$, $csc\theta = \frac{r}{y}$, $cot\theta = \frac{x}{y}$

Note also $tan\theta = \frac{sin\theta}{cos\theta}$.

Angle θ is measured in radians, $360° = 2\pi$ radians. $cos^2\theta + sin^2\theta = 1$

$sin(\theta + 2\pi) = sin\theta$
$cos(\theta + 2\pi) = cos\theta$
$sin(-\theta) = -sin\theta \neq sin\theta$
$cos(-\theta) = cos\theta \neq -cos\theta$

$cos\theta = \frac{x}{r} \Rightarrow x = rcos\theta$
$sin\theta = \frac{yr}{\Rightarrow} y = rsin\theta$
distance around the circle is $2\pi r \rightarrow 2\pi$ radians
distance halfway around is $\pi r \rightarrow \pi$ radians
distance quarter way around is $\frac{1}{2}\pi r \rightarrow \frac{\pi}{2}$ radians
length of arc corresponding to θ radians is $r\theta$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$d^2 = (coss - cost)^2 + (sins - sint)^2$
$= (cos(s-t) - 1)^2 + (sin(s-t))^2$
Hence $cos(s-t) = coss.cost + sins.sint$
and $cos(s+t) = coss.cost - sins.sint$
$cos2t = cos^2t - sin^2t$
$= 2cos^2t - 1 = 1 - 2sin^2t$
Replacing s by $\frac{\pi}{2} - s$ we get
$cos(\frac{\pi}{2} - s - t) = cos(\frac{\pi}{2} - s)cost + sin(\frac{\pi}{2} - s)sint$
$= cos(\frac{\pi}{2} - (s+t))$
so $sin(s+t) = sins.cost + coss.sint$
Hence $sin(s-t) = sins.cost - coss.sint$
$\rightarrow sin2t = 2sint.cost$
$[d^2 = cos^2s - 2cos.cost + cos^2t + sin^2s - 2sins.sint + sin^2t = cos^2(s-t) - 2cos(s-t) + 1 + sin^2(s-t)$Henc

Example: In the distance-velocity application the velocity is the rate of change of distance. The second derivative is the rate of change of the velocity which is the acceleration.

Example: If $f(x) = sin(x)$ then $f''(x) = -sin(x)$ and if $f(x) = cos(x)$ then $f''(x) = -cos(x)$. Thus $sin(x)$ and $cos(x)$ are solutions of the differential equaion $\frac{d^2y}{dx^2} = -y$.

Chapter 4

Derivatives of trigonometric functions

Let $y = sinx$. Then $\frac{dy}{dx}$

$$= \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h} = \lim_{h \to 0} \frac{sin(x)cos(h) + cos(x)sin(h) - sin(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{sin(x)(cos(h) - 1)}{h} + \frac{cos(x)sin(h)}{h} \right] = sin(x) \lim_{h \to 0} \frac{cos(h) - 1}{h} + cos(x) \lim_{h \to 0} \frac{sin(h)}{h}$$

We now proceed to show that

$$\lim_{h \to 0} \frac{cos(h) - 1}{h} = 0$$

and

$$\lim_{h \to 0} \frac{sin(h)}{h} = 1$$

.

The line PQ has length $2sin(h)$ and the arc PQ has length $2h$. Thus $sin(h) < h$. The triangle OTR has area $\frac{1}{2}tan(h)$ and the sector OSR has area $\frac{h}{2\pi}.\pi = \frac{1}{2}h$. Thus $h < tan(h)$. We conclude that $cos(h) < \frac{sin(h)}{h} < 1$ and hence that

$$\lim_{h \to 0} \frac{sin(h)}{h} = 1$$

.

Next we observe that $(1 - cos(h))(1 + cos(h)) = 1 - cos^2(h) = sin^2(h) < h^2$. Noting that everything is positive (taking $h > 0$) we can divide through by h(1+ cos(h)) to get $0 < \frac{1-cos(h)}{h} < \frac{h}{1+cos(h)}$ and thus

$$\lim_{h \to 0} \frac{1 - cos(h)}{h} = 0$$

. For $cos(h)-1$ or for negative h the inequalities change but the limit is still 0.

We conclude that $\frac{dy}{dx} = cos(x)$ (for $y = sin(x)$). Now for $y = cos(x)$, a similar argument gives

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{cos(x+h)-1}{h} = cos(x) \lim_{h\to 0} \frac{cos(h)-1}{h} - sin(x) \lim_{h\to 0} \frac{sin(h)}{h} = -sin(x)$$

.

We now introduce the second derivative, or derivative of the derivative:- if $y = f(x)$, then $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

4.1 Examples

1. $u = x^3$, $v = x^2$, $uv = x^5$ so $\frac{d}{dx}(uv) = 5x^4$. Now $u\frac{dv}{dx} + v\frac{du}{dx} = x^3.2x + x^2/3x^2 = 5x^4$.

2. If $f(x) = x.sin(x)$ then $\frac{du}{dx} = x.cos(x) + sin(x)$.

3. If $f(x) = sin^2(x)$ then we can take $u = v = sin(x)$ and get $f'(x) = sin(x)cos(x) + sin(x)cos(x) = 2sin(x)cos(x)$. Similarly, $\frac{d}{dx}(cos^2(x)) = -2cos(x)sin(x)$ (so that $\frac{d}{dx}(sin^2(x) + cos^2(x)) = 0$ as we would expect).

4. $\frac{d}{dx}(sec(x)) = \frac{d}{dx}(\frac{1}{cos(x)} = -\frac{1}{cos^2(x)}(-sin(x)) = \frac{sin(x)}{cos^2(x)} = sec(x)tan(x)$
   $\frac{d}{dx}(csc(x)) = \frac{d}{dx}(\frac{1}{sin(x)} = -\frac{1}{sin^2(x)}(cos(x)) = -\frac{cos(x)}{sin^2(x)} = -csc(x)cot(x)$

5. $\frac{x^5}{x^3} = \frac{x^3(5x^4)-x^5(3x^2)}{x^6} = \frac{5x^7-3x^7}{x^6} = 2x$ (as expected!)

6. $\frac{d}{dx}(tan(x)) = \frac{d}{dx}\left(\frac{sin(x)}{cos(x)}\right) = \frac{cos(x)cos(x)-sin(x)(-sin(x))}{cos^2(x)} = \frac{1}{cos^2(x)} = sec^2(x)$
   $\frac{d}{dx}(cot(x)) = \frac{d}{dx}\left(\frac{1}{tan(x)}\right) = -\frac{1}{tan^2(x)}.sec^2(x) = -\frac{1}{sin^2(x)} = -csc^2(x)$

7. $y = \frac{sin(x)+cos(x)}{sin(x)-cos(x)}$: $\frac{dy}{dx} = \frac{(sin(x)-cos(x))(cos(x)-sin(x))-(sin(x)+cos(x))(cos(x)+sin(x))}{(sin(x)-cos(x))^2}$
   $= \ldots = \frac{-2cos^2(x)}{(sin(x)-cos(x))^2}$.

Chapter 5

Rules for differentiation (1)

We have already seem the *sum rule* $\frac{d}{dx}(u(x)+v(x)) = \frac{du}{dx} + \frac{dv}{dx}$ and the *multiplication by a constant* rule $\frac{d}{dx}(cu(x)) = c\frac{du}{dx}$, for any constant c. These are usually combined into a *linearity rule* (for the derivative of a linear combination of functions):

$$\frac{d}{dx}(au(x) + bv(x)) = a\frac{du}{dx} + b\frac{dv}{dx}$$

The next usefule rule is for a *product* of two functions. If $f(x) = u(x)v(x)$ then

$$f'(x) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h))v(x+h) - v(x)) + v(x)(u(x+h) - u(x))}{h}$$

$$= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right]$$

$$= u(x)\frac{dv}{dx} + v(x)dudx$$

.

Thus we have $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Now consider a *reciprocal function* $\frac{1}{v(x)}$. Since $v(\frac{1}{v}) = 1$, we know that the derivative of $v(\frac{1}{v}) = 0$. Thus $v\frac{d}{dx}(\frac{1}{v} + \frac{1}{v}\frac{dv}{dx} = 0$ and hence $\frac{d}{dx}(\frac{1}{v}) = -\frac{1}{v^2}\frac{dv}{dx}$.

This allows us to formulate the more general *quotient rule* for a function $\frac{u(x)}{v(x)}$: we have $\frac{d}{dx}(\frac{u}{v}) = \frac{d}{dx}(\frac{1}{v}u) = \frac{1}{v} - u(\frac{1}{v^2}\frac{dv}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

5.1 Examples

1. $y = x^2 - 2x$ goes from negative slope to zero slope to positive slope. Note that this behaviour is also followed by $y = x^2 - 2x + 1$ and $y = x^2 - 2x + 5$. Positive slope does not mean "positive function".

2. In the diagram on the right the slope is $+0-0+0-0+$. From the graph we see that $f'(x) = u(x)(x-1)(x-2)(x-3)(x-4)$.

3. Suppose (for simplicity) that I can enter a straight motorway at any point. If ordinary driving speed is $30mph$ and motorway driving speed is 60mph where should I enter in order to minimise my journey time?

   Problem: minimise $\frac{\sqrt{a^2+x^2}}{30} + \frac{(b-x)}{60} = f(x)$. Here $f'(x) = \frac{1}{30}\frac{1}{2}(a^2 + x^2)^{-\frac{1}{2}}(2x) - \frac{1}{60} = \frac{2x-\sqrt{a^2+x^2}}{60(a^2+x^2)^{\frac{1}{2}}}$. Squaring both sides this gives $4x^2 = a^2 + x^2 and 3x^2 = a^2$. Thus $x = \pm\frac{a}{sqrt3}$. Clearly, $x = -\frac{a}{\sqrt{3}}$ doesn't make sense, so only $a = \frac{a}{\sqrt{3}}$ is valid. In fact, $f'(x) \neq 0$ when $x = -\frac{a}{\sqrt{3}}$! Then $f(\frac{a}{\sqrt{3}}) = \frac{(\sqrt{3}ab)}{60}$.

   Also, $f(0) = \frac{a}{30} + \frac{b}{60}$ and $f(b) = \frac{\sqrt{a^2+b^2}}{30}$. Clearly, $\frac{\sqrt{3}a}{60} < \frac{a}{30}$.
   So, $\frac{\sqrt{3}a+b}{60} < \frac{\sqrt{a^2+b^2}}{30}$ iff $b > \frac{a}{\sqrt{3}}(i.e. iff (a-\sqrt{3}b)^2 = 0)$.
   So optimal x is the smaller of $\frac{a}{\sqrt{3}}$ and b.

   We may also consider this from another point of view. If θ is the angle then the distance to the motorway is $asec\theta$ and the distance on the motorway is $b - atan\theta$. Thus, the time is $g(\theta) = \frac{asec\theta}{30} + \frac{b-atan\theta}{60}$. Now $g'(\theta) = \frac{a}{30}sec\theta tan\theta - \frac{a}{60}sec^2\theta$. Setting this equal to 0 and simplifying we get $sin\theta = \frac{1}{2}$. So to minimise the journey time we need to set $\theta = 30°$.

   NB: we did not check that the value of x or θ giving $f'(x) = 0$ or $g'(\theta) = 0$ corresponded to a minimum value of $f(x)$ or $g(x)$.

Chapter 6

Applications of the derivative

Consider the function $f(x)$ on the interval $[a,b]$. We say $f(x)$ is *increasing* if for any $a \leq x_1 < x_2 \leq b$ we have $f(x_1) < f(x_2)$ and *decreasing* is defined similarly. We have if $\frac{df}{dx} > 0$ then f(x) is increasing, if $\frac{df}{dx} < 0$ then f(x) is decreasing.

Suppose thaqt f(x) has a maximum value in the internal $[a,b]$. This must occur either at an *endpoint* or at a "*corner*" or at a *stationary point* where $\frac{df}{dx} = 0$. These are the three types of *critical point* (A "corner" is roughly speaking a point where $\frac{df}{dx}$ does not exist).

Considering values of x close to a stationary point we see that at a *local maximum point* the derivative goes from + 0 - so the function is increasing and then decreasing. Similarly, at a *local minimum point* the derivative goes - 0 +.

6.1 Examples

Find stationary points, corners, and endpoints and decide whether each point is a local or absolute maximum or minimum.

1. $f(x) = x^2 + \frac{2}{x}, 1 \leq x \leq 4$. $f'(x) = 2x - \frac{2}{x^2}$. $f'(x) = 0$ where $2x = \frac{2}{x^2}$ i.e. $x^3 = 1$. This is true at $x = 1$ and y division $x^3 - 1 = (x-1)(x^2 - x + 1)$. Now the quadratic factor has $b^2 - 4ac = 1 - 4 = -3$ so no real roots. Thus $x = 1$ is the only real value giving a stationary point and we have $f(1) = 1 + 2- = 3$. $f(0.9) \approx 3.03$, $f(1.1) \approx 3.03$ hence this is a local minimum. Now $f(4) = 16.5$. Thus $(1,3)$ is an absolute minimum and $(4, 16.5)$ is an absolute maximum.

2. $f(x) = (x - x^2)^2$, $-1 \leq x \leq 1$. $f(x) = x^2 - 2x^3 + x^4$. $f'(x) = 2x - 6x^2 + 4x^3 = 2x(1 - 3x + 2x^2) = 2x(1-2x)(1-x)$. $f'(x) = 0$ at $x = 0$, $x = \frac{1}{2}$, $x = 1$. Values on either side of $x = 0$ are positive so that gives a local minimum. $f(0.5) = 0.625$. $f(0.4) = f(0.6) = 0.24^2 \approx 0.06$ so $x = 0.5$

gives a local maximum. $f(1) = 0$. $f(0.9) = 0.0081$, $f(1.1) = 0.0121$ so $x = 1$ gives a local minimum. $f(-1) = 4$ so $x = -1$ gives the absolute maximum.

3. An airline will carry a box if length + width + height $= l + w + h \leq 158cm$. If h is fixed then $w = 158 - h - l$ so $V = l(158 - h - l)h$. Now $\frac{dV}{dl} = 158h - h^2 - 2hl$ so $\frac{dV}{dl} = 0$ at $l = \frac{158-h}{2}$. Then $V_{max} = \left(\frac{158-h}{2}\right)^2 h = 6241h - 79h^2 + \frac{1}{4}h^3$. Thus $\frac{dV}{dh} = 6241 - 158h + \frac{3}{4}h^2 = \frac{3}{4}(h - 158)(h - \frac{158}{3})$. Thus V is maximised by $h = \frac{158}{3}$ and then $h = w = l$ and the box is a cube.

6.2 More Examples

1. Suppose that four corners are cut from a square of cardboard of side $12cm$ and the sides folded up to make an open box. What is the maxiimum volume of the box?

   $V = (12 - 2x)^2 x = 144x - 48x^2 + 4x^3$

   $V = 0$ at $x = 0$ and $x = 6$.

   $\frac{dV}{dx} = 144 - 96x + 12x^2$. $\frac{dV}{dx} = 0$ where $x^2 - 8x + 12 = 0$.

   $(x - 2)(x - 6) = 0$

   $x = 2$ or $x = 6$.

   Clearly, the value $x = 6$ is not correct. Now one way to decide whether x=2 represents the maximum value is to observe that $\frac{dV}{dx} = 12(x-2)(x-6)$ and $\frac{dV}{dx}(1.9) > 0$ while $\frac{dV}{dx}(2.1) < 0$.

2. A fixed wall makes one side of a rectangle and we have $200m$ of fence for the other three sides. What is the maximum area that can be enclosed?

   $A = (200 - 2x)x$
   $= 200x - 2x^2$
   $\frac{dA}{dx} = 200 - 4x$
   $\frac{dA}{dx} = 0$ at $x = 50m$.

   Since $A(0) = A(100) = 0$ and $\frac{dA}{dx} = 4(50 - x)$ is positive for $x < 50$ and negative for $x > 50$ we conclude that $x = 50$ gives the maximum value $A = 500$.

3. A piece of wire $4m$ long is cut into two pieces, one of which forms a circle and the other a rectangle. Find the dimensions for min/max area.

   Length of circle is $2\pi r = 4 - 4x$ so $r = \frac{2-2x}{\pi}$ and total area is $A = \pi r^2 + x^2 = \pi\left(\frac{2-2x}{\pi}\right)^2 + x^2 = \left(\frac{4}{\pi} + 1\right)x^2 - \frac{8}{\pi}x + \frac{4}{\pi}$. Hence $\frac{dA}{dx} = 2(\frac{4}{\pi} + 1)x - \frac{8}{\pi}$ so $\frac{dA}{dx} = 0$ at $x = \frac{4}{4+\pi}$. Since the slope of $\frac{dA}{dx}$ is positive its value is negative to the left of $\frac{4}{4+\pi}$ and positive to the right, and hence $\frac{dA}{dx} = 0$ gives the minimum area. Note that radius for minimum area is $r = \frac{2}{4+\pi}$

and $A_{min} = \frac{4}{4+\pi} m^2$ (!). Also, this means that the maximum value of the area must be at an endpoint. Noting that $0 \leq x \leq 1$ we have $A(0) = \frac{4}{\pi}$, $A(1) = 1$ so the maximum area occurs where all the wire is used to make the circle.

6.3 Still more examples

1. Suppose $f(x) = sin(x)$ and $g(x) = x^2$. Then $(f \circ g)(x) = sin(x^2)$ and $(g \circ f)(x) = (sin(x))^2 = sin^2(x)$.
2. If $f(x) = x^3$ and $g(x) = x^4$ then $(g \circ f)(x) = (f \circ g)(x) = x^{12}$. But note that $f(x)g(x) = x^7$.
3. If $f(x) = x$ and $g(x) = x$ then $f \circ g)(x) = (g \circ f)(x) = x$. *This is the identity function.*
4. If $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$ then $(f \circ g)(x) = x - 1$, $(g \circ f)(x) = \sqrt{x^2 - 1}$.
5. If $f(x) = x + 5$, $g(x) = x - 5$ then $f \circ g)(x) = x = (g \circ f)(x)$. Thus the functions f and g are *inverse functions.*

6.4 Even more examples

1. $\frac{d}{dx}(sin(x^2)) = (cos(x^2))(2x) = 2xcos(x^2)$ $\frac{d}{dx}(sin^2 x) = (2sinx)(cosx = 2sinxcosx$
2. $\frac{d}{dx}((x^3)^4) = 4(x^3)^3(3x^2) = 12x^1 1$ $\frac{d}{dx}((x^4)^3) = 3(x^4)^2(4x^3) = 12x^1 1$
3. $\frac{d}{dx}(\sqrt{x^2 - 1}) = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 - 1}}$
4. $\frac{d}{dx}((x^3 + 1)^5) = 5(x^3 + 1)^4 .3x^2 = 15x^2(x^3 + 1)^4$
5. $\frac{d}{dx}((1 - x)^2) = 2(1 - x)(-1) = -2(1 - x)$
6. $\frac{d}{dx}(sin\sqrt{1 - x^2}) = (cos\sqrt{1 - x^2})(\frac{1}{2}(1 - x^2)^{-\frac{1}{2}})(-2x) = \frac{xcos\sqrt{1-x^2}}{\sqrt{1-x^2}}$
7. $\frac{d^2}{dx^2}(sinx^2) = \frac{d}{dx}(2xcosx^2) = 2cosx^2 - 4x^2 - 4x^2 sinx^2$

Chapter 7

Composite functions and the chain rule

If we think of a function as a black box into which we put a value of x and out of which we obtain a value of y, then it is easy to see that this process can be repeated to form a *composite function.*

Here we have $y = f(x)$ and $z = g(y) = g(f(x))$. The composite function that takes as input the value of x and gives the output z is denoted by $g \circ f$ (read "g after f") and we write $z = (g \circ f)(x)$. To differentiate $z = (g \circ f)(x)$ we need to consider a value of x of the form $x + \delta x$. This produces a change in y to $y + \delta y$ and a consequent change in z to $z + \delta z$. We want to find $\lim_{\delta x \to 0} \frac{\delta z}{\delta x}$ and the key is to write $\frac{\delta z}{\delta x} = \frac{\delta z}{\delta y} . \frac{\delta y}{\delta x}$. Now, as δx tends to zero, $\frac{\delta y}{\delta x}$ approaches $\frac{dy}{dx}$ and consequently δy must also be approaching 0 and then $\frac{\delta z}{\delta y}$ approaches $\frac{dz}{dy}$. Thus we have

$$\lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \lim_{\delta y \to 0} \frac{\delta z}{\delta y} \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

or $\frac{dz}{dx} = \frac{dz}{dy} . \frac{dy}{dx}$
or $g'(x(x))f'(x)$.

The chain of functions may be extended to more than two - the essential point is to differentiate from the outside (last function) in (first function)! Often other letters are used e.g. $y = sinu(t)$ has derivate $\frac{dy}{dt} = (cosu(t))u'(t)$.

Chapter 8

Derivative tests

8.1 The first derivative test

We saw how it is possible to distinguish between local maxima and local minima by considering values on either side of the stationary point. We can also do this by considering the sign of the tangent on either side since for a local maximum the tangent changes sign from positive to negative and for a local minimum the tangent changes sign from negative to positive.

Example: $y = sin(x)$ at $x = \frac{\pi}{2}$. $\frac{dy}{dx} = cos(x)$ goes from > 0 to < 0 as x increases.

Recall: an increasing function is indicated by a positive derivative; a decreasing function is indicated by a negative slope.

8.1.1 Examples

1. $f(x) = x^2 - 6x$, $f'(x) = 2x - 6$, $f'(x) = 0$ at $x = 3$, $f''(x) = 2 > 0$ so $x = 3$ gives a local minimum (as expected).

2. $f(x) = x^3 - 6x^2$, $f'(x) = 3x^2 - 12x$, $f'(x) = 0$ at $x = 0, x = 4$, $f''(x) = 6x - 12$, $f''(0) = -12 < 0$, $f''(4) = 12 > 0$ so at $x = 0$ we have a local maximum, at $x = 4$ we have a local minimum.

3. $f(x) = sin(x) - cos(x)$, $0 \leq x \leq 2\pi$, $f(x) = cos(x) + sin(x)$, $f''(x) = -sin(x) + cos(x)$. $f'(x) = 0$ at values of x where $cos(x) = -sin(x)$ or $tan(x) = -1$ so at $x = \frac{3}{4}\pi and \frac{7}{8}\pi$. $f''(\frac{3}{4}\pi) = -\sqrt{2}$ gives a maximum value, $f''(\frac{7}{8}\pi) = \sqrt{2}$ gives a minimum.

4. $f(x) = x + x^2 - x^3$, $f'(x) = 1 + 2x - 3x^2 = (1 + 3x)(1 - x)$, $f'(x) = 0$ at $x = -\frac{1}{3}, 1$. $f''(x) = 2 - 6x$, $f''(-\frac{1}{3}) > 0$ so this is a local minimum; $f''(1) < 0$ so this is a local maximum. Hence there is an infection at $f''(x) = 0$ where $x = \frac{1}{3}$.

5. $f(x) = (x-2)^2(x-4)^2$, $f'(x) = 4(x-2)(x-4)(x-3) = 4(x^3 - 9x^2 + 28x - 4)$, $f''(x) = 4(3x^2 - 18x + 28)$. $f'(x) = 0$ at $x = 2, 3, 4$. $f''(2) > 0$ local minimum, $f''(3) < 0$ local maximum, $f''(4) > 0$ local minimum. Inflections at $f''(x) = 0$, $x = 3 \pm \frac{\sqrt{3}}{3}$

8.2 The second derivative test

It follows that at a local maximum point the second derivative is negative and at a local minimum point the second derivative is positive.

Points at which the curve y changes from concave up to concave down or from concave down to concave up are called *inflection points.* At such points the second derivative goes from negative to positive or positive to negative, or in other words, crosses the axis, i.e. has value 0.

However f'' can also be zero at local maximum and local minmum points so when we obtain $f'' = 0$ we must check further using the derivative or values of the function or a sketch of the function.

8.2.1 Examples

1. If the position x of a body on a line at time t is given by $x = \sqrt{3t^2+4}$ find the velocity and speed, and the acceleration at $t = 2sec$. Here $V = \frac{dx}{dt} = \frac{1}{2}(3t^2+4)^{-\frac{1}{2}}(6t) = \frac{3t}{\sqrt{3t^2+4}}$ and $a = \frac{d^2x}{dt^2} = ... = \frac{12}{(\sqrt{3t^2+1})^3} = \frac{12}{(3t^2+1)^{\frac{3}{2}}}$. Thus $V(2) = \frac{6}{\sqrt{16}} = \frac{3}{2}$, $a(2) = \frac{12}{16^{\frac{3}{2}}} = \frac{12}{64} = \frac{3}{16}$, speed $= |\frac{3}{2}| = \frac{3}{2}$.

2. A ball is thrown straight up in the air from a height of $2m$, with an initial velocity of $19.6m/sec$ at $t = 0$. Find (a) the height y above the earth's surface, as a function of t; (b) the velocity at time t; (c) the maximum height the ball reaches; (d) the time elapsed until the ball hits the ground. Here $y = 2 + 19.6t - 4.9t^2$ and $\frac{dy}{dt} = V = 19.6 - 9.8t$. At the instant of maximum height the ball stops so its velocity is zero. This occures where $19.6 - 9.8t = 0$ i.e. at $t = 2sec$. The maximum height is therefore $y = 2 + (19.6)(2) - (4.9)(2)^2 = 21.6m$. Finally, the ball reaches the ground when $2 + 19.6t - 4.9t^2 = 0$ i.e. where $t = -0.1sec$ or $4.1sec$. Excluding the negative root (which corresponds to the extrapolation backwards in time of the flight of the ball to the point where it wouldhave left the ground) we have $t = 4.1sec$.

8.2.2 Examples

1. Suppose $y^5 + xy = 3$. Using the chain rule and the product rule we obtain $5y^4\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$. Thus $(5y^4 + x)\frac{dydx}{=} - y$ or $\frac{dy}{dx} = \frac{-y}{5y^4+x}$. Thus we can find the value of $\frac{dy}{dx}$ for any point (x, y) where $5y^4 + x \neq 0$. For example $(2, 1)$ lies on the curve and $\frac{dy}{dx}|_{(2,1)} = -\frac{1}{7}$.

2. Suppose $sin(x) + sin(y) = 1$. Then $cosx + cosy\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{cos(x)}{cos(y)}$ where $cos(y) \neq 0$.

3. Suppose $ysiny = x$. Then $ycosy\frac{dy}{dx} + siny\frac{dy}{dx} = 1$ and we can solve for $\frac{dy}{dx}$ if necessary.

4. Find the tangent direction to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. Here $2x + 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$ and $\frac{dy}{dx}|_{(3,4)} = -\frac{3}{4}$.

Chapter 9

Application of the second derivative: acceleration

If s is the distance from the origin of an objhect then its *velocity* is the rate of change of s, or $\frac{ds}{dt}$, and the rate at which the velocity changes is the *acceleration* given by $\frac{d^2s}{dt^2}$. We use the term *speed* for the magnitude of the velocity (i.e. ignoring the direction) so speed is $|v|$.

If an object is moving vertically near the earth's surface influenced only by gravity (and ignoring air resistance) it can be shown that its height y is given by an equation of the form $y = y_0 + v_0 t - \frac{1}{2}gt^2$ where y_0, v_0, g are constants. When $t = 0$, we see that $y = y_0$ so y_0 is the initial height at time $t = 0$. Also, $v = \frac{dy}{dx} = v_0 - gt$ so $v(0) = v_0$ and hence v_0 is the initial velocity. Finally, $a = \frac{d^2y}{dx^2} = -g$ so $-g$ is the *acceleration due to gravity*, which is approximately $9.8m/sec^2$.

Chapter 10

Implicit differentiation and related rates

When x and y are relation by an equation of the form $y = f(x)$ we can differentiate to find $\frac{dy}{dx}$. However, in some situations we cannot express y as a function of x explicitly; instead y is related implicitly to x by an equation of the form $g(x, y) = 0$. We then use the techniques of implicit differentiation to find $\frac{dy}{dx}$.

Often we have informqation on one rate of change and need to find a different rate of change. This is achieved by usin the chain rule. The essential idea is to obtain an equation relating the variables whose rates are known or required.

10.1 Examples

1. Sides of rectangle grow so that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. What is the value of $\frac{dx}{dt}$ at the instant when $x = 4$, $y = 3$. Here $x^2 + y^2 = z^2$ so $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$ and hence $2x\frac{dx}{dt} + \frac{2}{3}y\frac{dx}{dt} = 2z$ which on substituting $x = 4$, $y = 3$ gives $\frac{dx}{dt}$ =???.

2. Wen $y = 50$ find (a) the rate of change of z; (b) the rate of change of area; (c) the rate of change of θ. First note that $z = 50\sqrt{5}$ at $y = 50$.
(a) $z^2 = y^2 + 100^2$ so $2z\frac{dz}{dt}???2y\frac{dy}{dt}$. Thus at $(50, 50\sqrt{5})$, $\frac{dz}{dt} = \frac{3.50}{50\sqrt{5}} = \frac{3\sqrt{5}}{5}m/s$.
(b) $A = \frac{1}{2}(100)y = 50y$ so $\frac{dA}{dt} = 150m^2/s$.
(c) $sin\theta = \frac{y}{z}$ so $cos\theta\frac{d\theta}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2}$ and since $cos\theta|_{y=50} = \frac{100}{50\sqrt{5}}$ we can solve $\frac{d\theta}{dt} = \frac{3}{125}rad/s$.

3. A person $2m$ tall walks directly away from a streetlight that is $8m$ above the ground. If the person's shadow is lengthening at a rate of $\frac{4}{9}m/s$, at what rate in m/s is the person walking?

We are given $\frac{ds}{dt} = \frac{4}{9}$ and want to find $\frac{dx}{dt}$. By similar triangles $\frac{x}{6} = \frac{s}{2}$ s $x = 3s$. Hence $\frac{dx}{dt} = 3\frac{ds}{dt} = \frac{4}{3}m/s$.

10.2 Examples of max/min problems

1. Find two numbers whose sum is 6 and whose product is as large as possible. Let x, y be the numbers so $x + y = 6$ and we want to maximise $P = xy$. We have $y = 6 - x$ so $P = x(6 - x)$ and $\frac{dP}{dx} = 6 - 2x$ is zero at $x = 3$. There is no minimum value so $x = y = 3$ and $P = 9$. Note the correspondence with the pro blem of finding a rectangle of given perimeter and maximum area.

2. Find the point on the line $x + 3y = 6$ closest to $(-3, 1)$.
   $s = \sqrt{(x+3)^2 + (y-1)^2}$ is maximum where s^2 is maximum. Using $x = 6 - 37$ we get $s^2 = (9 - 3y)^2 + (y - 1)^2 = \frac{ds^2}{dy} = 2(9 - 3y)(-3) + 2(y - 1) = 20y - 56$ is zero at $y = \frac{14}{5}$, $x = -\frac{12}{5}$. The problem has a solution, so the point is $(-\frac{12}{5}, \frac{14}{5})$.

3. A manufacturer makes cylindrical cans for packaging food. Find the ratio of height to radius to minimise the amount of material used, assuming the same material is used for the ends and the curved surface.
   Here $S = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$ and $h = \frac{V}{\pi r^2}$.
   Hence $S = 2\pi r^2 + 2\pi r(\frac{V}{\pi r^2}) = 2(\pi r^2 + \frac{V}{r})$.
   Thus $\frac{dS}{dt} = 2(2\pi r - \frac{V}{r^2})$ is zero where $2\pi r = \frac{V}{r^2}$ or $r^3 = \frac{V}{2\pi}$. The problem clearly has a solution so this is it. We are interested in $\frac{h}{r}$ so consider $= \pi r^2 h$, $h = \frac{V}{\pi r^2}$, $\frac{h}{r} = \frac{V}{\pi r^3}$. Least material is used when $r^3 = \frac{V}{2\pi}$ so when $\frac{h}{r} = \frac{V}{\pi(\frac{V}{2\pi})} = 2$ i.e. when height = diameter.